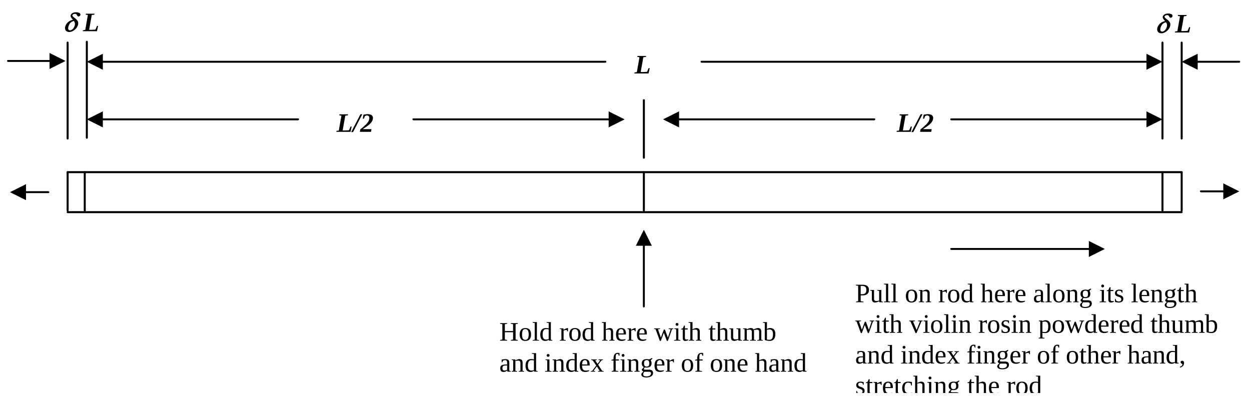
The Physics of a Longitudinally Vibrating "Singing" Metal Rod:

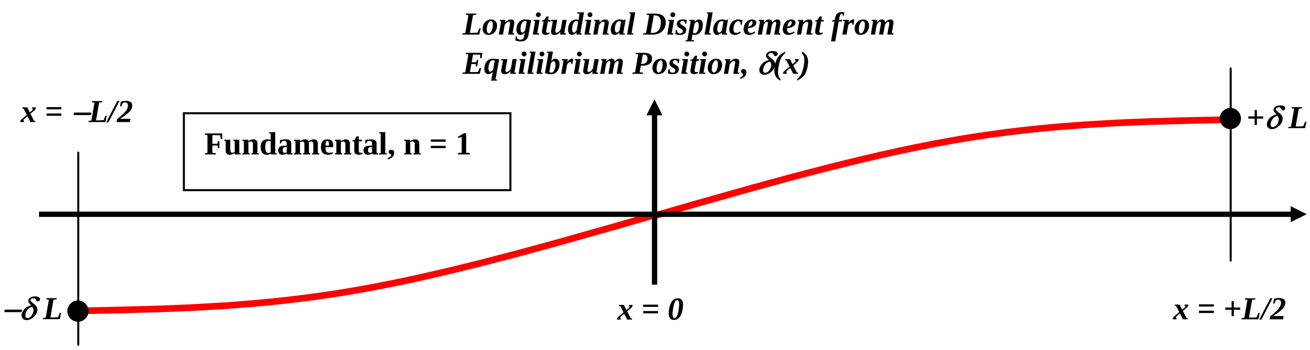
A metal rod (e.g. aluminum rod) a few feet in length can be made to vibrate along its length – make it "sing" at a characteristic, resonance frequency by holding it precisely at its mid-point with thumb and index finger of one hand, and then pulling the rod along its length, toward one of its ends with the thumb and index finger of the other hand, which have been dusted with crushed violin rosin, so as to obtain a good grip on the rod as it is pulled.

The pulling motion of the thumb and index finger actually stretches the rod slightly, giving it

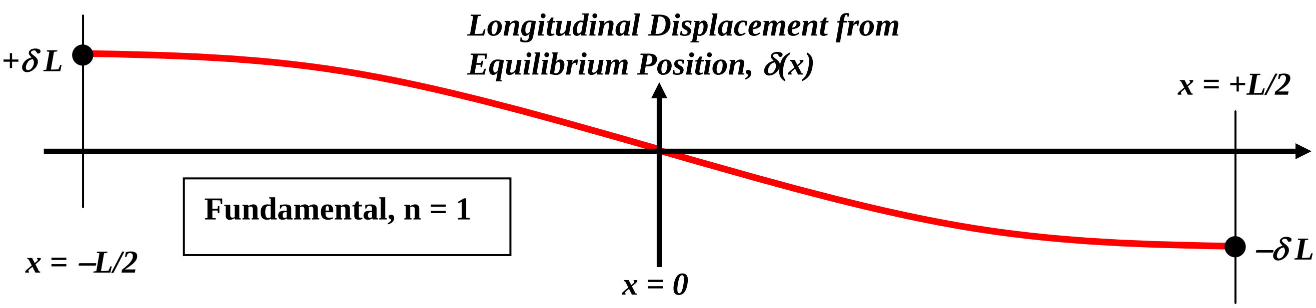


potential energy – analogous to the potential energy associated with stretching a spring along its length, or a rubber band. The metal rod is actually an *elastic* solid – elongating slightly when pulled! Pulling on the rod in this manner *excites* the rod, causing *both* of its ends to simultaneously vibrate longitudinally, back and forth along its length at a characteristic resonance frequency known as its fundamental frequency, f_1. For an excited aluminum rod of length, L ~ 2 meters, it is thus possible that at one instant in time both ends of the rod will be extended a small distance, δL ~ 1 mm beyond the normal, (i.e. non-stretched) equilibrium position of the ends of the rod. At another instant, one-half cycle later, the ends or the rod are compressed inwards this same amount. The displacement amplitude is a maximum at the ends of the rod. A point along the vibrating rod where the displacement amplitude is a maximum is known as a displacement *anti-node*. Thus both ends of the rod are displacement anti-nodes.

This type of excitation of a metal rod is known as the so-called fundamental, or first harmonic (n = 1) mode of excitation, or vibration – because this mode of vibration has the *lowest* possible frequency of vibration. The rod of length L vibrates in its fundamental mode with one-half of a wavelength, i.e. $L = ½\,\lambda_1$. The longitudinal displacement from equilibrium, along the length of the rod, as a function of position, is shown in the figure below. It can be seen that at the mid-point of the rod, the displacement amplitude is zero for this mode of vibration of the rod. A point along the vibrating rod where the displacement amplitude is zero is known as a *node*. This is why the rod is held precisely at its mid-point. If the rod is held near to, but not at its mid-point, this mode of excitation of the rod is much harder to accomplish, and it is also quite rapidly damped out – the vibrational energy that is present in the rod is absorbed in one's hand where it is held. Precisely at the mid-point of the rod, there is no net displacement at that point, hence there is no way energy can be transferred from the rod to one's hand; thus the rod "sings" for a very long time, gradually decaying away from energy loss associated with direct radiation of sound waves into the air, and internal frictional dissipation processes associated with the finite stiffness of the rod.



One half cycle later in the oscillation of the rod, the longitudinal displacement would appear as:



A further one half cycle later, the longitudinal displacement will again be as shown in the top picture above, and so on, as time progresses. At half-way times in between these two moments, the longitudinal displacement from equilibrium position is momentarily zero everywhere along the rod. The fundamental mode of vibration of the rod has one node, at its mid-point, $x = 0$.

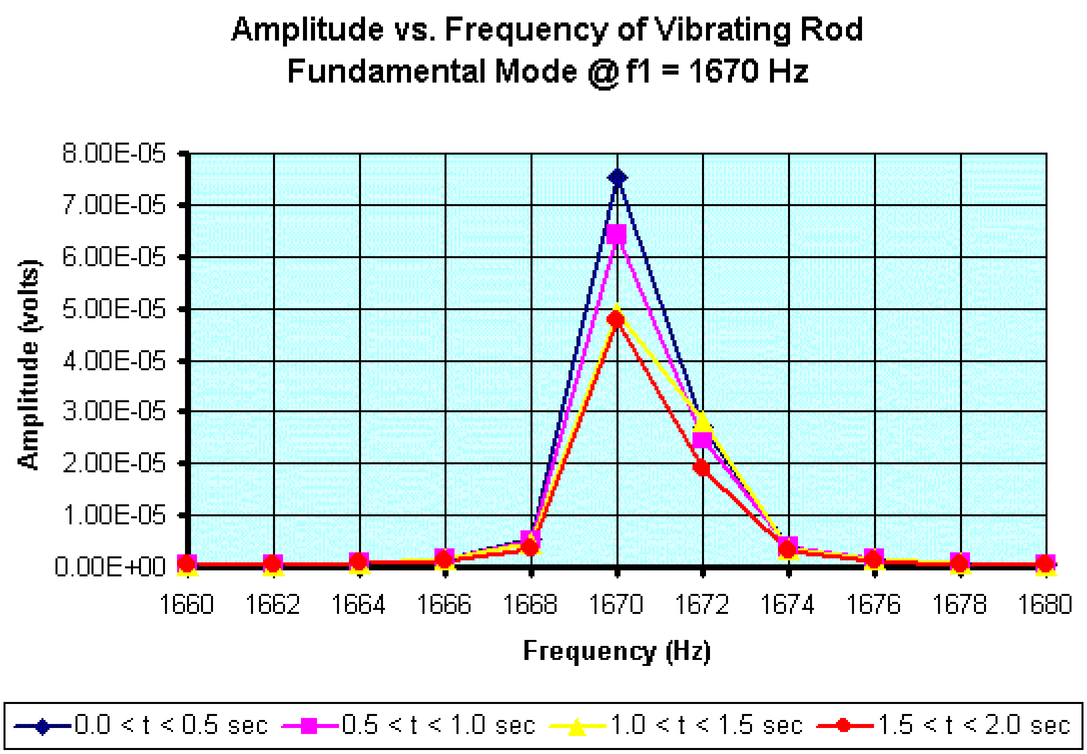
The (longitudinal) speed of propagation of sound in the metal rod, v is given by the formula: where $Y = \sigma/\varepsilon$ = Young's modulus, also known as the *tensile elastic* modulus. It is the ratio of

$$v = \sqrt{\frac{Y}{\rho}}$$

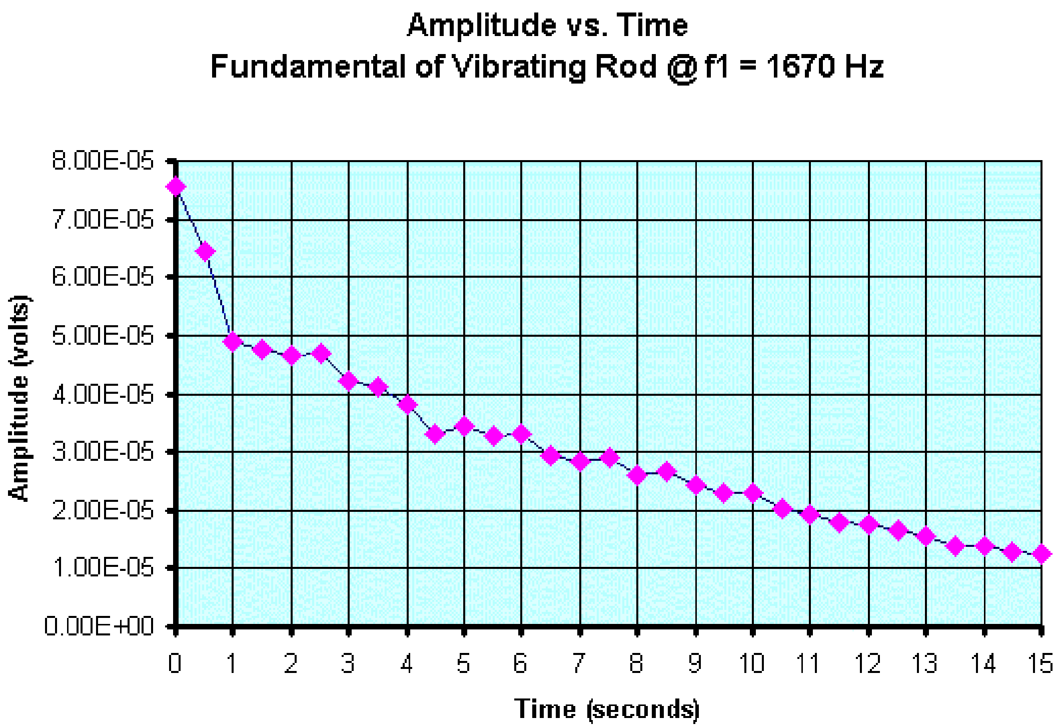
longitudinal, compressive stress, $\sigma = F/A$ (longitudinal compressive force per unit cross sectional area of the rod) to the longitudinal compressive strain, $\varepsilon = |L_2\text{-}L_1|/L_1$ where L_1 is the equilibrium length of the rod, and L_2 is the extended length of the rod when stretched. The density (mass per unit volume) of the rod, is denoted by ρ. Aluminum has a density of ρ_{AL} = 2.71 gm/cm^3 = 2710 kg/m^3 and has a Young's modulus of $Y_{AL} = 70\times10^9$ N/m^2. Thus, the speed of sound in aluminum rod is thus v_{AL} = 5082.4 m/s.

For an aluminum rod measured to be L = 1.52 meters long, the fundamental mode of vibration corresponds to a wavelength of $\lambda_1 = 2L$ = 3.04 meters. From the relationship between propagation speed, frequency and wavelength, namely that $v = f_1\lambda_1$, then for the fundamental mode of vibration of the aluminum rod, we thus have $f_1 = v/\lambda_1$ = (5082.4 m/s)/(3.04 m) = 1671.8 Hz (cycles per second).

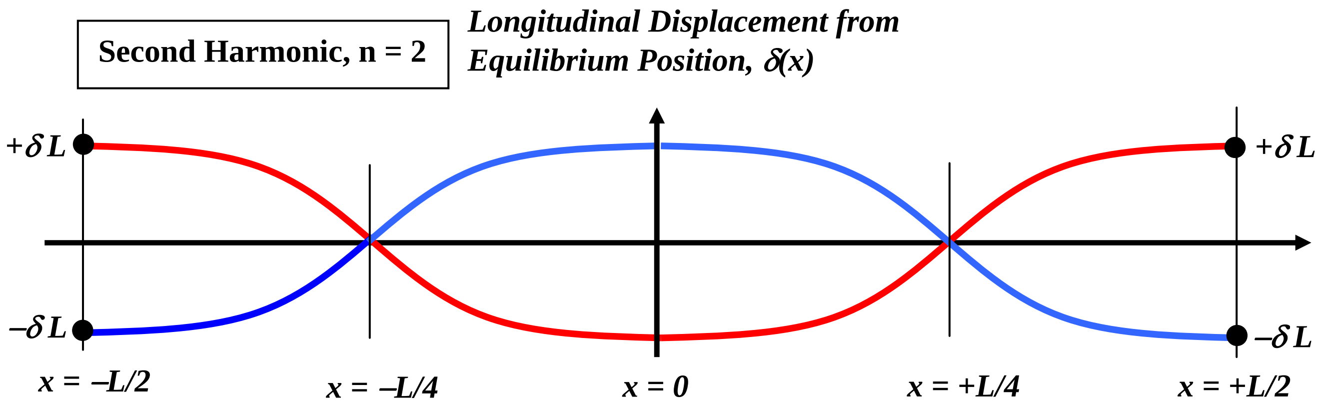
Using a PC-based data acquisition system consisting of a National Instruments PC+ DAQ card and LabView DAQ software, we recorded 15 seconds of the signal waveform from a microphone held in proximity to the aluminum rod, vibrating in its fundamental mode. The DAQ program carried out so-called a Fast Fourier Transform/Harmonic Analysis of the recorded waveform, in 30 one-half second time-slices. The measured frequency of the fundamental mode of vibration was f_1 ~ 1670±2 Hz, in good agreement with the above theoretical prediction. We show the results for the first four time slices (i.e. first two seconds) of the 15 second duration measurement in the figure below:



The fundamental frequency of the vibrating rod actually has a finite width of ~ 2 Hz. It can be seen that as time increases, the amplitude decreases. The following figure shows the amplitude signal output from the microphone, recording the fundamental mode of vibration as a function of time. It can be seen that initially there is a relatively quick decrease, followed by a less rapid decrease in sound output with time. Thus, this plot indicates that there are at least two types of energy loss mechanisms associated with the vibrating rod – internal dissipation in the rod, the other, sound radiation into the air!



It is also possible to excite other, higher modes of vibration of the rod. Instead of holding the rod at its mid-point, one can hold the rod at a point one-quarter of its length, measured from one end of the rod. Pulling on the rod along its length with rosin-dusted thumb and index fingers of the free hand will excite the next higher, second harmonic mode (n = 2) with a frequency, $f_2 = 2 f_1$ = 2*1671.8 Hz = 3343.6 Hz. This corresponds to a wavelength, $\lambda_2 = v/f_2$ = (5082.4 m/s)/(3343.6 Hz) = L = 1.52 meters. The displacement from equilibrium along the length of the rod, for this higher mode of oscillation, would thus appear as:

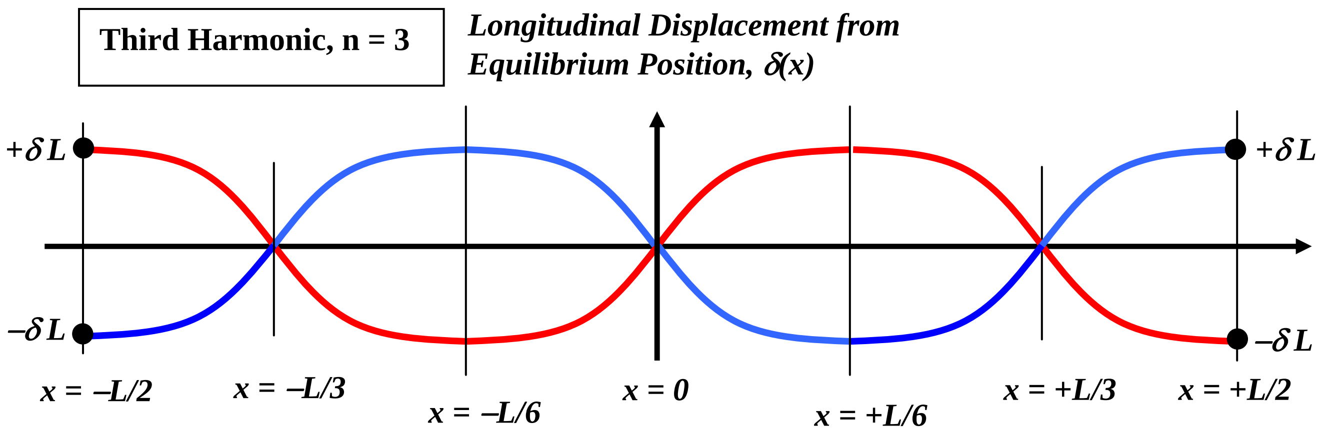


The red curve is the longitudinal displacement profile, δ(x) along the rod at one instant in time, say at time t = 0 seconds. The blue curve is the longitudinal displacement profile, d(x) along the rod one half cycle of oscillation later, at time $t = \tau_2/2$, where $\tau_2 = 1/f_2$ is the period of for this mode of vibration of the rod. The frequency f_2 is twice that of the fundamental frequency, f_1, (i.e.

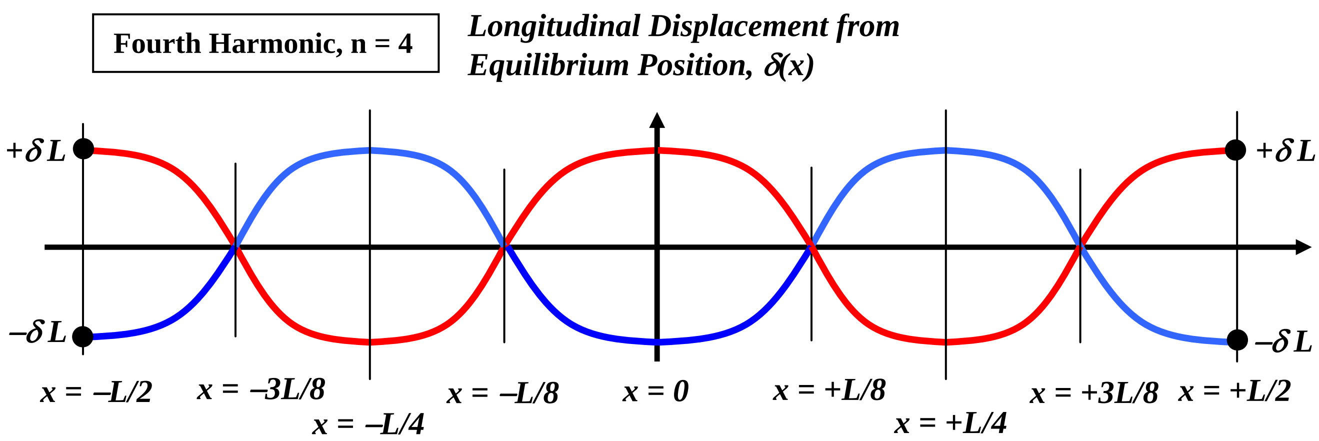
one octave above) since the wavelength, $\lambda_2 = L$ for this mode of vibration of the rod is half that of the wavelength, $\lambda_1 = 2L$ associated with the fundamental mode. This mode of vibration of the rod has two nodes, located at $x = \pm{}^1/_4\, L$ and three anti-nodes, one located at the mid-point of the rod at x = 0, and at the two ends of the rod, at $x = \pm{}^1/_2\, L$.

The next higher, third harmonic mode of vibration of the rod ($n = 3$) is shown in the figure below. The frequency f_3 is three times higher than that of the fundamental frequency, f_1, since the wavelength, $\lambda_3 = {}^2/_3 L$ for this mode of vibration of the rod is one third of that of the wavelength, $\lambda_1 = 2L$ associated with the fundamental mode. This mode of vibration of the rod has three nodes, one node located at $x = 0$, and two others located at $x = \pm{}^1/_3\, L$. This mode of vibration has four anti-nodes, two located at $x = \pm{}^1/_6\, L$ and two located at the ends of the rod, at $x = \pm{}^1/_2\, L$.

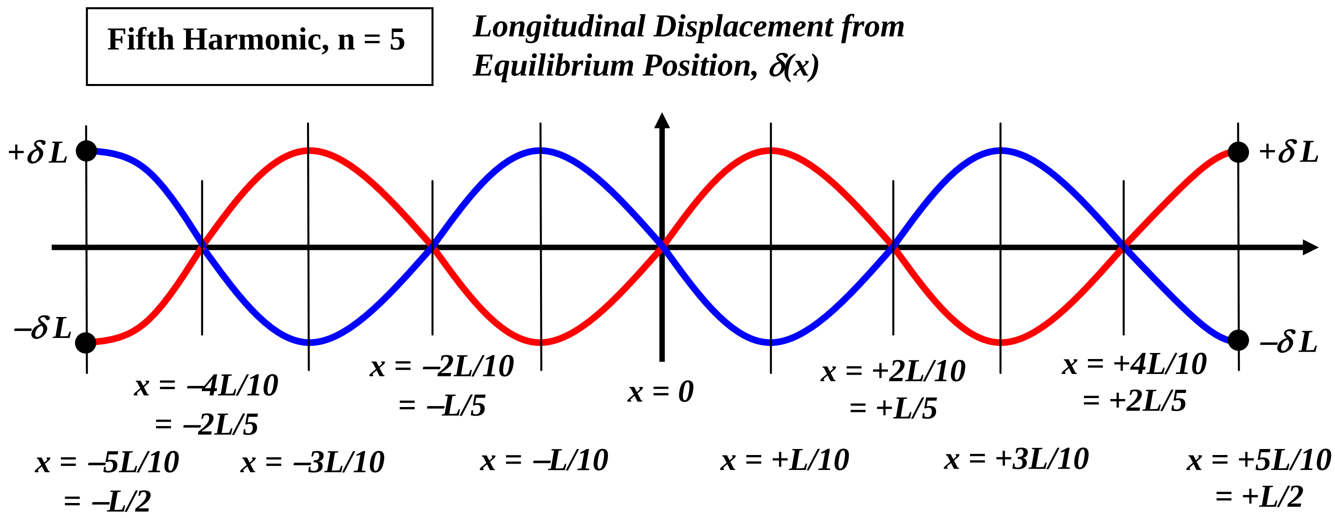
The next higher, fourth harmonic mode of vibration of the rod ($n = 4$) is shown in the figure



below. The frequency f_4 is four times (i.e. two octaves) higher than that of the fundamental frequency, f_1, since the wavelength, $\lambda_4 = {}^2/_4 L = ½\, L$ for this mode of vibration of the rod is one fourth of that of the wavelength, $\lambda_1 = 2L$ associated with the fundamental mode. This mode of vibration of the rod has four nodes, two nodes located at $x = \pm{}^1/_8\, L$, and two others located at $x = \pm{}^3/_8\, L$. There are five anti-nodes, one located at $x = 0$, two located at $x = \pm{}^1/_4\, L$ and two located at the endpoints, at $x = \pm{}^1/_2\, L$.



The next higher, fifth harmonic mode of vibration of the rod (n = 5) is shown in the figure below. The frequency f_5 is five times higher than that of the fundamental frequency, f_1, since the wavelength, $\lambda_4 = {}^2/_4 L = ½\, L$ for this mode of vibration of the rod is one fourth of that of the wavelength, $\lambda_1 = 2L$ associated with the fundamental mode. This mode of vibration of the rod has five nodes, one at $x = 0$, two nodes located at $x = \pm{}^1/_5\, L$, and two others located at $x = \pm{}^2/_5\, L$. There are six anti-nodes, two located at $x = \pm{}^1/_{10}\, L$, two located at $x = \pm{}^3/_{10}\, L$ and two located at the endpoints, at $x = \pm{}^5/_{10}\, L = \pm{}^1/_2\, L$.



There in fact exists an infinite hierarchy of so-called *normal modes of vibration* of the rod. Note that all modes (n = 0, 1, 2, 3, 4, 5, …) of vibration of the rod all have the *same* longitudinal speed of propagation of sound in the rod,

$$v = f_1\lambda_1 = f_2\lambda_2 = f_3\lambda_3 = ... = f_n\lambda_n$$

the frequencies of the higher modes are integer multiples of the fundamental mode, $f_n = n\, f_1$, where n = 1, 2, 3, 4, 5, …. The wavelengths associated with the higher modes of vibration are related to the wavelength of the fundamental mode by $\lambda_n = \lambda_1/n = 2L/n$.

When a person excites the rod by holding the rod at its mid-point with one hand and pulling on it with rosin-dusted thumb and index fingers of the other hand, not only the fundamental is excited, but in fact the third, fifth, seventh, ninth, … – all odd-n harmonics (n = 1, 3, 5, 7, 9,….) are also excited. Note that the odd harmonics all have a node at the mid-point of the rod, $x = 0$, where it is held.

If the rod is held at $x = \pm{}^1/_4\, L$ to excite the 2nd harmonic, it can be seen that this location is at an anti-node of the 4th harmonic – thus the 4th harmonic cannot be simultaneously excited by holding the rod at this point. Only if harmonics simultaneously have a common node at a given location along the length of the rod, will it then be possible to simultaneously excite more than one such harmonic of the rod.

We summarize the modal frequencies, wavelengths, and the locations of nodes and anti-nodes for the first nine harmonics associated with a vibrating rod of length, L in the table below.

Harmonic Mode #, n	**Frequency f (Hz)**	**Wavelength λ (m)**	**Node Locations, (m)**	**Anti-Node Locations, (m)**
1	f_1	$\lambda_1 = 2L$	0	$\pm{}^1/_2L$
2	$f_2 = 2f_1$	$\lambda_2 = {}^1/_2\lambda_1 = L$	$\pm{}^1/_4L$	$0, \pm{}^2/_4L = \pm{}^1/_2L$
3	$f_3 = 3f_1$	$\lambda_3 = {}^1/_3\lambda_1 = {}^2/_3L$	$0, \pm{}^2/_6L = \pm{}^1/_3L$	$\pm{}^1/_6L, \pm{}^3/_6L = \pm{}^1/_2L$
4	$f_4 = 4f_1$	$\lambda_4 = {}^1/_4\lambda_1 = {}^1/_2L$	$\pm{}^1/_8L,\ \pm{}^3/_8L$	$0, \pm{}^2/_8L = \pm{}^1/_4L, \pm{}^4/_8L= \pm{}^1/_2L$
5	$f_5 = 5f_1$	$\lambda_5 = {}^1/_5\lambda_1 = {}^2/_5L$	$0, \pm{}^2/_{10}L = \pm{}^1/_5L, \pm{}^4/_{10}L = \pm{}^2/_5L$	$\pm{}^1/_{10}L,\ \pm{}^3/_{10}L, \pm{}^5/_{10}L = \pm{}^1/_2L$
6	$f_6 = 6f_1$	$\lambda_6 = {}^1/_6\lambda_1 = {}^1/_3L$	$\pm{}^1/_{12}L,\ \pm{}^3/_{12}L = \pm{}^1/_4L, \pm{}^5/_{12}L$	$0, \pm{}^2/_{12}L = \pm{}^1/_6L, \pm{}^4/_{12}L= \pm{}^1/_3L, \pm{}^6/_{12}L= \pm{}^1/_2L$
7	$f_7 = 7f_1$	$\lambda_7 = {}^1/_7\lambda_1 = {}^2/_7L$	$0,\pm{}^2/_{14}L = \pm{}^1/_7L, \pm{}^4/_{14}L = \pm{}^2/_7L, \pm{}^6/_{14}L = \pm{}^3/_7L$	$\pm{}^1/_{14}L,\ \pm{}^3/_{14}L, \pm{}^5/_{14}L, \pm{}^7/_{14}L = \pm{}^1/_2L$
8	$f_8 = 8f_1$	$\lambda_8 = {}^1/_8\lambda_1 = {}^1/_4L$	$\pm{}^1/_{16}L,\ \pm{}^3/_{16}L, \pm{}^5/_{16}L, \pm{}^7/_{16}L$	$0, \pm{}^2/_{16}L = \pm{}^1/_8L, \pm{}^4/_{16}L= \pm{}^1/_4L, \pm{}^6/_{16}L= \pm{}^3/_8L$
9	$f_9 = 9f_1$	$\lambda_9 = {}^1/_9\lambda_1 = {}^2/_9L$	$0,\pm{}^2/_{18}L = \pm{}^1/_9L, \pm{}^4/_{18}L = \pm{}^2/_9L, \pm{}^6/_{18}L = \pm{}^3/_9L= \pm{}^1/_3L, \pm{}^8/_{18}L = \pm{}^4/_9L$	$\pm{}^1/_{18}L,\ \pm{}^3/_{18}L, \pm{}^5/_{18}L, \pm{}^7/_{18}L, \pm{}^9/_{18}L = \pm{}^1/_2L$

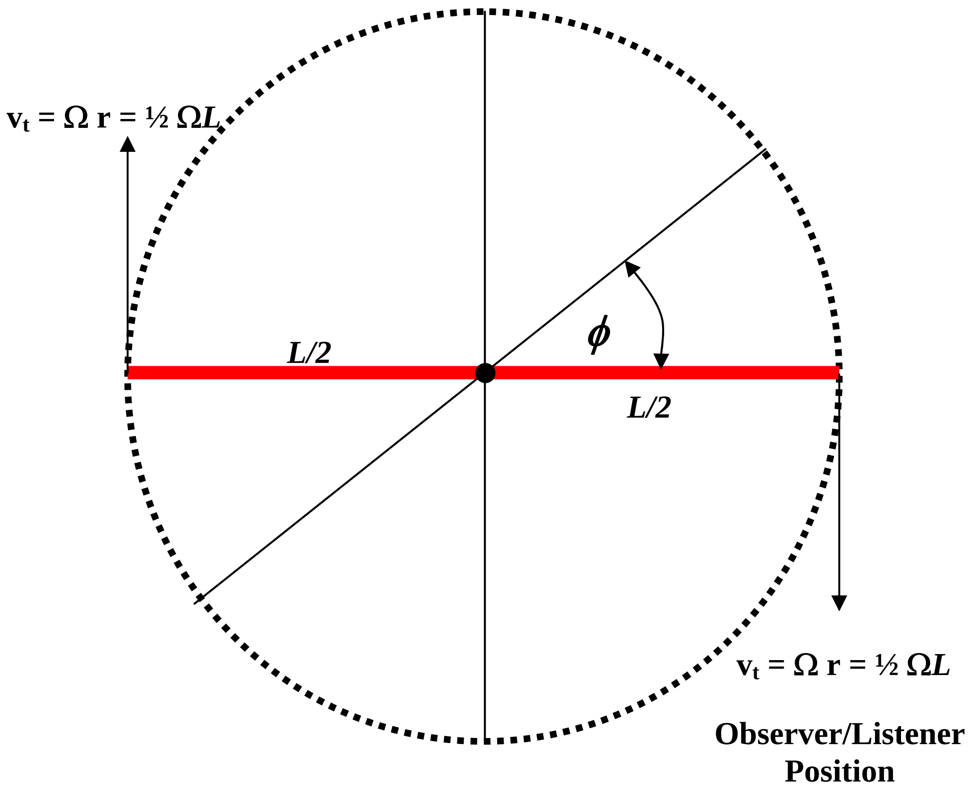
We used a Hewlett-Packard HP-3652A Dynamic Signal Analyzer (another piece of electronic measurement equipment in our lab) to carry out real-time Fast-Fourier Transform/Harmonic Analysis, in order to measure the harmonic content of the modal vibrations of our L = 1.52 m long aluminum rod. When the aluminum rod was held at its mid-point and excited, as expected, we observed the first three of the odd-harmonic (n = 1, 3, 5) modes of vibration of the rod, which we measured to be at f_1 = 1672 Hz, f_3 = 5012 Hz (~ 3 f_1) and f_5 = 8350 Hz (~ 5 f_1), respectively. Nearly all of the vibrational energy of the aluminum rod (> 99%) is in the fundamental – very little energy is contained in the higher (n = 3, n = 5) harmonics. Most of this is in the 3rd harmonic, with even less in the 5th harmonic. Higher order (n = 7, 9, …) harmonics were not observable with our setup.

When the rod was held at x = ¼ L in order to excite (only) the 2nd harmonic, we observed (only) the second (n = 2) harmonic at f_2 = 3338 Hz (~ 2 f_1).

When the rod was held at $x = {}^1/_3$ L, in order to excite (only) the 3rd harmonic odd harmonic, we observed (only) the third (n = 3) harmonic at f_3 = 5012 Hz (~ 3 f_1).

Motional Effects associated with a Rotating, Vibrating Rod: Doppler Effect and Beats

If the aluminum rod of length, L is excited by holding it at its mid-point and then rotated in a manner similar to that of twirling a baton, as shown in the figure below, one hears a warbling tone instead of the usual steady tone, because of the so-called Doppler effect, in honor of the German scientist who first discovered this motional effect.



The rotating ends of the vibrating rod are moving sound sources. If the (angular) frequency of rotation of the rod is $\Omega = 2\pi f_{rot}$ radians per second, this corresponds to a rotational frequency of $f_{rot} = \Omega/2\pi$ revolutions per second, or a period of $\tau_{rot} = 1/f_{rot} = 2\pi/\Omega$ seconds per revolution. The end of the rotating rod that is moving toward a listener is Doppler-shifted to a higher frequency. Conversely, the end of the rotating rod that is moving away from the listener is Doppler-shifted to a lower frequency. The ends of the rod are rotating at a tangential speed of $v_t = \Omega r = ½\Omega L$.

For a stationary observer/listener, the formula for the Doppler-shifted frequency, f' in terms of the original frequency, f, the speed of sound, v and the speed of the moving source, v_s is given by:

$$f' = \left(\frac{v}{v \pm v_s}\right) f$$

where the + sign is associated with the sound source moving directly away from the listener, and the – sign is associated with the sound source moving directly toward the listener. This configuration occurs at only at two points during the rotation cycle of the rod. At other times, the rod is oriented at an angle, $\phi = \Omega t$, as shown in the above figure (note that we define the zero of time such that at $t = 0$, $\phi = 0$). Then we can write the Doppler shift formula for the frequency associated with each end of the rod as:

$$f_1' = \left(\frac{v}{v - v_s \cos\phi}\right) f = \left(\frac{v}{v - v_s \cos\Omega t}\right) f$$

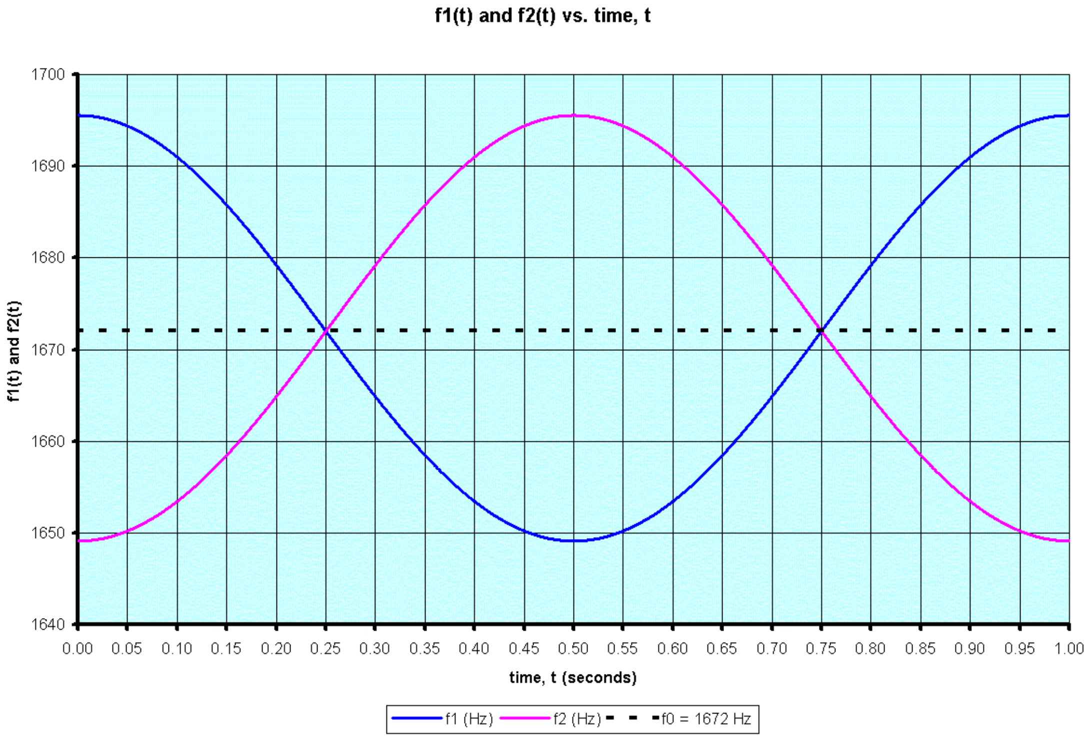
for the end (# 1) of the rod moving directly towards the observer/listener at t = 0 (when $f_1' > f$) and:

$$f_2' = \left(\frac{v}{v + v_s \cos\phi}\right) f = \left(\frac{v}{v + v_s \cos\Omega t}\right) f$$

for the end (# 2) of the rod moving directly away from the observer/listener at t = 0 (when $f_2' < f$). Thus, the rotating vibrating rod actually emits two time-dependent frequencies, one slightly higher and one slightly lower than the original frequency, f for a stationary, non-rotating rod. Because of the small difference in frequency between f_1' and f and f_2' and f, the two Doppler-shifted frequencies, f_1' and f_2' are also similar to each other.

When two sounds are superimposed upon each other that differ from each other only slightly in frequency, the resultant, overall sound is one which is equivalent to a sound which has the average of the two frequencies, $\langle f_{avg} \rangle = ½ (f_1' + f_2')$ but which is *amplitude modulated* at the *difference* frequency, $\Delta f = | f_1' - f_2'|$. This acoustical phenomenon is known as beats, and is shown in the figures below, for two rotating sound sources as given by the above expressions for the two time-dependent frequencies, f_1' and f_2' for a rotating vibrating aluminum rod, of length L = 1.52 meters, fundamental frequency f = 1672 Hz, rotating at frequency of $f_{rot} = 1$ revolution/second, with rotational period $\tau_{rot} = 1/ f_{rot} = 1$ second.

At a rotational frequency of $f_{rot} = 1$ revolution/second, the maximum (minimum) Doppler-shifted frequencies are ~ 23 Hz higher (lower) than the stationary, non-rotational fundamental (n = 1) frequency of f = 1672 Hz, respectively. Thus, the maximum difference frequency due to the rotational motion associated with the Doppler effect is $\Delta f = | f_1' - f_2'| \sim 46$ Hz, when the vibrating rod is *perpendicular* to the observer/listener's line of sight. However, as the vibrating rod rotates, the Doppler-shifted frequencies f_1' and f_2'also change with time. When the vibrating rod is oriented such that it is *parallel* to the line of sight of the observer/listener (this occurs at two times – at
$t = ¼\ \tau = 0.25$ sec and at $t = ¾\tau = 0.75$ sec), at those moments in time when there is no Doppler shift of either sound source (from the listener's perspective) and hence no beats are heard at that instant in time, since the two frequencies are identical, the beat frequency between them $\Delta f = | f_1' - f_2'| = 0$ Hz. When the rod is again oriented perpendicular to the observer/listener's line of sight (at $t = ½\ \tau = 0.5$ sec and at $t = \tau = 1.0$ sec), then the Doppler shifts high and low are again maximal, with the maximal beat frequency! The two frequencies, $f_1'(t)$ and $f_2'(t)$ as a function of time, t for one entire rotational period, are shown in the figure below.



If the individual displacement amplitudes associated with the sounds emanating from each of the two individual sound sources are given by:

$$y_1(x,t) = A_1 \cos(\omega_1 t) = A_1 \cos(2\pi f_1 t)$$

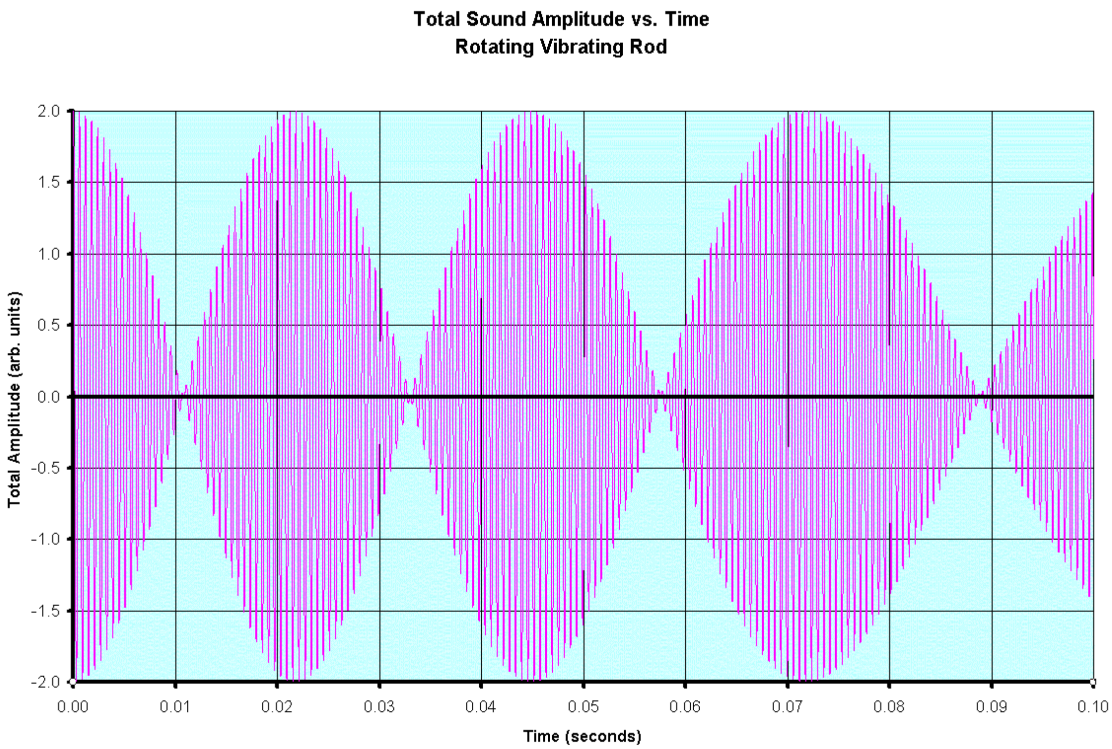
and

$$y_2(x,t) = A_2 \cos(\omega_2 t) = A_2 \cos(2\pi f_2 t)$$

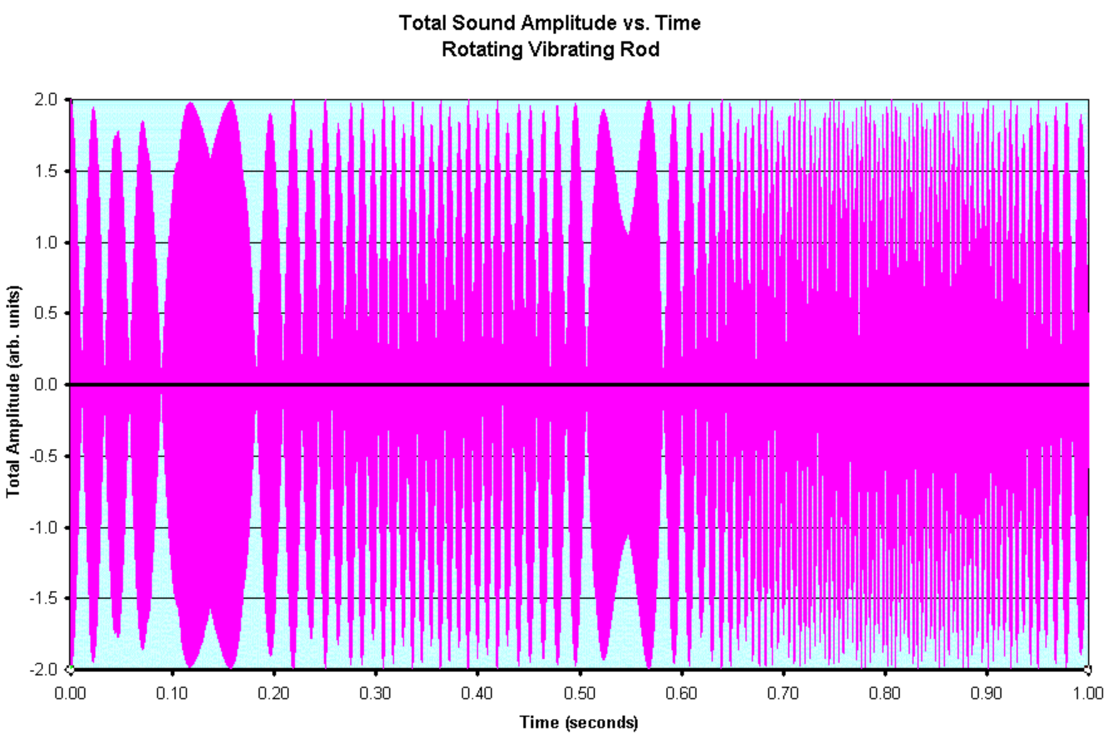
Then the total displacement amplitude is just the linear sum of the two individual amplitudes:

$$y_{tot}(x,t) = y_1(x,t) + y_2(x,t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t)$$

In the first figure below, we show the total displacement amplitude, $y_{tot}(x,t) = y_1(x,t) + y_2(x,t)$ for the first $^1/_{10}$ of a rotational period, τ so that the high-frequency structure associated with the two individual frequencies, f_1 and f_2 (~ 1670 ± 23 Hz) can be readily observed. The *envelope* of the high-frequency waveform is modulated at the beat frequency, $\Delta f = | f_1' - f_2' |$.



In the second picture, we show the total displacement amplitude, $y_{tot}(x,t) = y_1(x,t) + y_2(x,t)$ for one entire rotational period, $\tau = 1$ second. The beat pattern between the two sound sources has quite a lot of interesting structure, due to the rotational Doppler effect.



The actual beat pattern is in fact even more complicated than that shown above, due to the fact(s) that a.) the rotating vibrating rod is a spatially extended sound source, b.) the listener/observer is a finite distance from the vibrating rod, and c.) the acoustic environment also affects the overall sound. Sounds emanating from the ends of the rod are reflected off of walls, which the listener also hears, in addition to the sound waves coming directly from the two ends of the rotating vibrating rod.

Thus, we can see that quite a wide variety of acoustical phenomena can be observed associated with the longitudinal vibrations of a simple metal rod!

Musically, it is conceivable that an entire marching band could play a musical piece where each marching band member twirled their own vibrating aluminum rod of a given length, tuned to a given frequency. As we have discussed, twirling such vibrating rods like batons gives them an additional, rich sound texture due to the Doppler effect. A marching band playing a musical piece with twirling vibrating aluminum rods of varying lengths would make for a very unique half-time show e.g. at the Rose Bowl!

Would Mandi Patrick, UIUC Feature Twirler, be willing to lead such a marching band???

Also, if interested, check out e.g. **Tom Kaufmann** (musician and instrument builder) playing a "friction harp" – a whole collection of longitudinally-vibrating rods on YouTube:

http://www.youtube.com/watch?v=47wkiyLsc2U
and:
http://www.youtube.com/watch?v=g4i2mzQqNRY

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